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Renewal Sequences and Intermittency

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In this paper we examine the generating function $\Phi(z)$ of a renewal sequence arising from the distribution of return times in the "turbulent" region for a class of piecewise affine interval maps introduced by Gaspard and Wang and studied by several authors. We prove that it admits a meromorphic continuation to the entire complex z-plane with a branch cut along the ray $(1, +\infty)$. Moreover, we compute the asymptotic behavior of the coefficients of its Taylor expansion at $z=0$. From this, we obtain the exact polynomial asympotics for the rate of mixing when the invariant measure is finite and of the scaling rate when it is infinite.

KEY WORDS: Renewal sequence; intermittency; mixing rate; scaling rate; zeta function.

INTRODUCTION

The Pomeau–Manneville[(9)] type 1 intermittency model (at the tangent bifurcation point) consists of a class of smooth transformations $f: [0, 1] \to [0, 1]$ which are expanding everywhere but at a neutral fixed point at the origin. Such *intermittent interval maps* provide with no doubt the simplest examples of "chaotic" dynamical systems with anomalous statistical behaviour. For instance they may possess but a σ-finite non-normalizable invariant measure.[(10–12)] Or else, they may leave invariant a probability measure with slow (e.g., polynomial) speed of mixing.[(14–16)] Finally, in the framework of thermodynamic formalism, they may exhibit phase transitions[(6–8)] and dynamical ζ-functions with non-polar singularities.[(14, 18)]

Here we consider a one parameter family of linearized intermittent interval maps (see Eq. (1.4) below) introduced by Gaspard and Wang.[(1, 2)] As far as its statistical properties are concerned, it is equivalent to a

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Markov chain with countable state space.[13] It is also related to the statistical mechanics model introduced by Fisher[17] and successively studied by Gallavotti[18] (see also Hofbauer[19]). Without pretending to be a typical example, the main advantage of this approximation scheme is that it partially allows for exact calculations.

The main concern of this paper is the study of the generating function of a renewal sequence arising from the distribution of return times in a region where the map is uniformly expanding. This function coincides, up to a factor $(1-z)^{-1}$, with the Ruelle dynamical ζ-function, and is shown to admit a meromorphic continuation to the the entire complex z-plane with a branch cut along the ray $(1, +\infty)$. It appears that finding analytic continuation of dynamically defined functions, which are holomorphic in a domain given *a priori*, can be a rewarding mathematical achievement in itself. Moreover, it will be shown that the main statistical features of this dynamical system such as the rate of mixing or the scaling rate, are embodied in the behaviour of its Taylor coefficients.

1. PRELIMINARIES

Let q, r, s be three real parameters satisfying $0<q<1$, $s>0$ and $r+1=q^{-s}$. Let moreover $d_n \downarrow 0$ be the sequence defined by

$$d_0=1, \qquad d_n=(1+nr)^{-1/s}, \qquad n\geqslant 1 \tag{1.1}$$

In particular $d_1=q$. The sequence d_n is a Kaluza sequence, i.e., it satisfies:[20]

$$0<d_n\leqslant d_0=1, \qquad d_n^2\leqslant d_{n-1}d_{n+1} \tag{1.2}$$

which is equivalent to the assertion that $d_{n+1}-2\lambda d_n+\lambda^2 d_{n-1}\geqslant 0$ for all $\lambda>0$. The numbers d_n's generate a countable partition $\mathscr{A}$ of $[0, 1]$ into the intervals $A_n=[d_n, d_{n-1}]$, $n\geqslant 1$. Let $\rho_n:=m(A_n)=d_{n-1}-d_n$ (here and in the sequel m denotes the Lebesgue measure). One then readily verifies that $\rho_n<\rho_{n-1}$ and ρ_n/ρ_{n-1} is increasing. In addition, one has the asymptotic behaviour

$$\rho_n=(r/s)(1+nr)^{-1-(1/s)}+\mathscr{O}((1+nr)^{-2-(1/s)}) \tag{1.3}$$

Setting $A_0=[0, 1]$ we define $p_k=m(A_k)/m(A_{k-1})$, $k\geqslant 1$, and consider the piecewise affine map f: $[0, 1]\to[0, 1]$ defined by

$$f(x)=\begin{cases}(x-d_1)/p_1, & \text{if } \ x\in A_1\\ d_{k-1}+(x-d_k)/p_k, & \text{if } \ x\in A_k, \quad k\geqslant 2\end{cases} \tag{1.4}$$

This map has been introduced as as a simplified model for an intermittent interval map whose behaviour when $x \to 0^+$ is given by

$$f(x) = x + ux^{1+s} + \mathcal{O}(x^{1+s+\varepsilon})$$

where $u = r/s$ and $\varepsilon > 0$. The fixed point at the origin is neutral: $f'(0) = 1$, and $f'(x)$ is only Hölder continuous at $x = 0$, with exponent s. It may also be noted that if $\psi_0(x)$ denotes the inverse branch of f which maps $[0, 1]$ onto $[0, q]$, then we have $d_n = \psi_0^n(1)$, and the sequence d_n is Kaluza if and only if ψ_0 is concave. We finally observe that $f(A_n) = A_{n-1}$ for any $n \geqslant 1$, so that $\mathscr{A}$ is a Markov partition for the map f.

1.1. A Countable Markov Chain

One can say more: the iteration process $x_n = f^n(x)$, with f as above and x randomly chosen according to Lebesgue measure, is actually isomorphic (mod 0) to a Markov chain with state space $\mathbb{N}$ and transition matrix $P = (p_{ij})$ given by

$$P = \begin{pmatrix} \rho_1 & \rho_2 & \rho_3 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.5}$$

To see this, let X be the residual set of points in $(0, 1]$ which are not preimages of 1 with respect to the map f, namely $X = (0, 1] \backslash \{d_n\}_{n \geqslant 0}$. Let moreover Ω be the set of all one-sided sequences $\omega = (\omega_0 \omega_1 \cdots)$, $\omega_i \in \mathbb{N}$ s.t. given ω_i then $\omega_{i-1} = \omega_i + 1$ or $\omega_{i-1} = 1$. Then the map φ: $\Omega \to [0, 1]$ defined by

$$\varphi(\omega) = x \quad \text{according to} \quad f^j(x) \in A_{\omega_j}, \qquad j \geqslant 0$$

is a bijection between Ω and X and conjugates the map f with the shift T on Ω. It is then immediate to check that the stochastic process on Ω given by $x_i(\omega) = \omega_i$, $i \geqslant 0$, is a Markov chain with conditional probabilities $p_{ij} = P(x_n(\omega) = j \mid x_{n-1}(\omega) = i) = m(f^{-1}(A_j) \cap A_i)/m(A_i)$, which coincide with those in (1.5). Since g.c.d. $\{n : \rho_n > 0\} = 1$ the chain is aperiodic and recurrent. Consider the infinite sequence $t_1, t_2, \ldots$ of successive entrance times in the state 1: $t_1 = \inf\{i \geqslant 0 : \omega_i = 1\}$ and, for $j \geqslant 2$, $t_j = \inf\{i > t_{j-1} : \omega_i = 1\}$. Let moreover $r_j = t_{j+1} - t_j$ be the sequence of times between returns. The state 1 being recurrent, the numbers r_j are i.i.d.r.v. under the probability $P_1(\cdot)=P(\cdot \mid x_0(\omega)=1)$. Their common distribution is $P_1(r_j=n)=\rho_n$ and their expectation value is given by $E_1(r_j)=\sum n\rho_n = \sum d_n$, which may be finite (positive-recurrent chain) or infinite (null-recurrent chain) according whether $s<1$ or $s\geqslant 1$. More specifically, we can define a family of moments

$$M^{(\ell)}=E_1(r_j^\ell)\equiv \sum n^\ell \rho_n, \qquad \ell \geqslant 0 \tag{1.6}$$

and say that the chain has *ergodic degree* ℓ if $M^{(\ell)}<\infty$ but $M^{(\ell+1)}=\infty$. Notice that $M^{(0)}=1$, so that the chain has degree at least zero (null-recurrent case). Finally, the steady-state equation is $\pi_n=\sum_{i\in S}\pi_i p_{in}$ and is formally solved by $\pi_n=\pi_1 d_{n-1}$, $n\geqslant 1$. In the positive-recurrent case one finds $\pi_1=(\sum d_n)^{-1}$. For more details on this Markov chain we refer to Isola[13].

1.2. Invariant Measure and Return Times

An easy consequence of the previous discussion is that the map f preserves an absolutely continuous σ-finite measure ν, whose density e is given by

$$e(x)=\pi_n/(\rho_n\pi_1)=d_{n-1}/\rho_n, \qquad d_n<x\leqslant d_{n-1} \tag{1.7}$$

It may be noted that

$$\nu(A_n)=d_{n-1}=\sum_{l\geqslant n} m(A_l) \tag{1.8}$$

More specifically, for $E\subseteq A_n$ we have, using (1.7),

$$\nu(E)=\frac{m(E)}{m(A_n)}\,d_{n-1} \tag{1.9}$$

Let τ: $X\to\mathbb{N}$ be the *first passage time* in the interval A_1, that is

$$\tau(x)=1+\min\{n\geqslant 0 : f^n(x)\in A_1\} \tag{1.10}$$

so that A_n is the closure of the set $\{x\in X : \tau(x)=n\}$. On the other hand, the *return time* function r: $X\to\mathbb{N}$ in the interval A_1 is given by

$$r(x)=\min\{n\geqslant 1 : f^n(x)\in A_1\}=\tau\circ f(x) \tag{1.11}$$

Let B_n = closure of $\{x \in A_1 : r(x) = n\}$. Clearly $B_n = A_1 \cap f^{-1} A_n$, and therefore, using (1.9), we get

$$\nu(B_n) = \frac{m(A_1 \cap f^{-1} A_n)}{m(A_1)} = m(A_n) \tag{1.12}$$

as one can easily check. Putting togheter (1.8) and (1.12) we have the following chain of formal identities:

$$\begin{aligned} \nu([0, 1]) &= \sum \nu(A_n) = \sum n m(A_n) = \int_0^1 \tau(x)\, m(dx) \\ &= \sum_n n \nu(B_n) = \int_{A_1} r(x)\, \nu(dx) = M^{(1)} \end{aligned} \tag{1.13}$$

which is a version of Kac's formula. Clearly (1.13) becomes meaningful under the assumption that all terms involved are finite. Using the shorthand $M \equiv M^{(1)}$ we then have the following dichotomy: either $M < \infty$, and then there exists an f-invariant a.c. probability measure $\mu = \nu/M$; or $M = \infty$ so that ν is not normalizable and no invariant a.c. probability measure exists. In the latter case, the ergodic means $(1/n) \sum_{k=0}^{n-1} \delta_{f^k(x)}$ converge weakly to the Dirac delta at 0.[21, 11] For later use, we now define recursively a family of formal "tail sequences" $d_n^{(\ell)}$, with $\ell \geqslant 0$, derived from d_n as follows:

$$d_n^{(0)} = d_n \qquad \text{and} \qquad d_n^{(\ell)} = \sum_{l > n} d_l^{(\ell - 1)} \qquad \text{for} \quad \ell > 0 \tag{1.14}$$

Moreover we say that a_n and b_n are asymptotically equivalent as n approaches ∞, denoted as $a_n \sim b_n$ $(n \to \infty)$, if the quotient a_n / b_n tends to unity. From (1.1) (see also (1.3)) we have that if $s < 1/\ell$, with $\ell \geqslant 1$, then the terms $d_n^{(k)}$ are finite for $0 \leqslant k \leqslant \ell$ and satisfy

$$d_n^{(k)} \sim (1 + (n + k)\, r)^{k - (1/s)} \tag{1.15}$$

It is also easy to check that $M^{(\ell)}$ is finite if and only if $d_n^{(\ell)}$ is. Of special importance will be the asymptotic behaviour of $d_n^{(1)}$, when $s < 1$:

$$d_{n-1}^{(1)} = \nu(x \in X : \tau(x) > n) \sim C\, n^{1 - (1/s)} \tag{1.16}$$

where $C = r^{1 - (1/s)}$.

2. THE GENERATING FUNCTION OF THE RETURN TIMES DISTRIBUTION

If we view the element A_n of the countable Markov partition $\mathscr{A}$ introduced in Section 1 as the nth "state" for our dynamical system, the number ρ_n can be interpreted as the m-probability that a first passage in the state 1 occurs after n iterates. Let us now consider the quantity $u_n := m(f^{-(n-1)}A_1)$ which, for $n \geqslant 1$, gives the m-probability to observe a *passage* in the state 1 after $n-1$ iterates (for the first time or not). We can write

$$\begin{aligned} u_n &= \sum_{r=1}^{n} \rho(f^l(x) \notin A_1, 0 \leqslant l < r-1, f^{r-1}(x) \in A_1, f^{n-1}(x) \in A_1) \\ &= \sum_{r=1}^{n} \rho(\tau(x) = r)\, \rho(f^{n-1}(x) \in A_1 \mid \tau(x) = r) \\ &= \sum_{r=1}^{n} \rho_r \rho(f^{n-1}(x) \in A_1 \mid \tau(x) = r) \end{aligned}$$

On the other hand, according to the discussion given in the previous section, the iteration process $x_n = f^n(x)$ "starts afresh" at each passage in the state 1. This implies

$$\begin{aligned} \rho(f^{n-1}(x) \in A_1 \mid \tau(x) = r) &= \rho(f^{n-1}(x) \in A_1 \mid f^{r-1}(x) \in A_1) \\ &= \rho(f^{n-r}(x) \in A_1 \mid x \in A_1) \\ &= \rho(f^{n-r-1}(x) \in A_1) = u_{n-r} \end{aligned}$$

and therefore the sequence $u_0, u_1, \ldots$ satisfies the recurrence relation:

$$u_0 = 1 \qquad \text{and} \qquad u_n = \rho_n + u_1 \rho_{n-1} + \cdots + u_{n-1} \rho_1 \qquad \text{for} \quad n \geqslant 1 \tag{2.1}$$

In other words, $u_0, u_1, \ldots$ is the *renewal sequence*[20] associated with the sequence $\rho_1, \rho_2, \ldots$. We now show that u_n is also equal to $\nu(A_1 \cap f^{-n}A_1)$, and can thus be interpreted as the ν-probability to observe a *return* in the state 1 after n iterates (recall that $\nu(A_1) = 1$). Indeed, setting $u_n^{(1)} := \nu(A_1 \cap f^{-n}A_1)$ and reasoning as above, we see that $u_n^{(1)}$ satisfies the recurrence relation:

$$u_0^{(1)} = 1 \qquad \text{and} \qquad u_n^{(1)} = u_0^{(1)} \nu(B_n) + \cdots + u_{n-1}^{(1)} \nu(B_1) \qquad \text{for} \quad n \geqslant 1 \tag{2.2}$$

where B_n is defined after (1.11). On the other hand we know that $\nu(B_n) = m(A_n) \equiv \rho_n$ and, comparing with (2.1), we get $u_n^{(1)} \equiv u_n$, $\forall n$.

We now turn to the study of the generating function $\Phi(z)$ of the sequence u_n, which is given by

$$\Phi(z) = \sum_{n=0}^{\infty} u_n z^n = \left(1 - \sum_{n=1}^{\infty} \rho_n z^n\right)^{-1} = \left((1-z) \sum_{n=0}^{\infty} d_n z^n\right)^{-1} \tag{2.3}$$

The next result can be viewed as a sharpening of a renewal theorem proved by Erdös, Feller and Pollard.[22]

Theorem 2.1 (Part One). The power series defined in (2.3) defines a holomorphic function $\Phi(z)$ in the open unit disk and converges at every point of the unit circle with the exception of $z=1$, where it has a non-polar singular point. Moreover, one has the following asymptotic behaviour of the coefficients u_n: let $C_1 = \sin(\pi/s)/(\pi r^{1-(1/s)})$ and $C_2 = C_1/M$, then

(a) for $s<1$ we have $v_n := Mu_n - 1 \sim C_2 n^{1-(1/s)}$;

(b) for $s \geqslant 1$ we have

$$u_n \sim \begin{cases} C_1 n^{-1+(1/s)}, & \text{if} \quad s>1 \\ 1/\log n, & \text{if} \quad s=1 \end{cases}$$

Proof. We first notice that

$$0 \leqslant \frac{1}{\sum_{n=0}^{\infty} d_n} = \frac{1}{M} < 1$$

It is then easy to see that the function $D(z) := \sum_{n=0}^{\infty} d_n z^n$ has no zeros for $|z| \leqslant 1$. Indeed, for $|z|<1$ this follows from (2.3), since $\rho_n > 0$ and therefore $|\sum_{n=1}^{\infty} \rho_n z^n| < 1$ for $|z|<1$. Furthermore, from the above identity it follows that any zeros of $D(z)$ must be of the form $e^{i\phi}$, $0<\phi<2\pi$. Now, if $D(e^{i\phi})=0$ then (2.3) implies $\sum_{n=1}^{\infty} \rho_n e^{in\phi} = 1$, that is $\cos(n\phi)=1$, $\forall n \geqslant 1$, which is impossible. Then the function $1/D(z)$ has no singularities in $|z|<1$ and we can expand it in a power series $1/D(z) = \sum_{n=0}^{\infty} \gamma_n z^n$. Notice that $\gamma_0 = 1$. Set $h_n = -\gamma_n$ $(n \geqslant 1)$. We can then say more. By the property (1.2) of the sequence d_n we can aplly Hardy,[23] Theorem 22, and obtain

$$h_n \geqslant 0, \qquad \sum_{n=1}^{\infty} h_n \leqslant 1$$

In addition, if $M=\infty$, then $\sum_{n=1}^{\infty} h_n = 1$. In particular, it appears that $1/\sum_{n=0}^{\infty} d_n z^n$ is absolutely convergent for $|z| \leqslant 1$. This yields the announced analytic properties of $\Phi(z)$ (the nature of the singularity at $z=1$ will be clarified at the end of the proof).

To show statement (b), it may be noted that $u_n = 1 - h_1 - \cdots - h_n$ so that, if $M = \infty$, the sequence u_n decreases monotonically to 0. Assertion (b) then follows from (1.15) and a repeated application of a Tauberian theorem for power series (see, e.g., Feller,[24] Chap. XIII.5, Theorem 5).

Next, we are going to prove statement (a), for $s < 1$. In this case, we have $u_n \to 1/M$ as $n \to \infty$. To obtain more information we first note that the relation

$$\sum_{n=0}^{\infty} u_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n = \sum_{n=0}^{\infty} z^n$$

implies

$$\sum_{n=0}^{\infty} v_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n = \sum_{n=0}^{\infty} d_n^{(1)} z^n \tag{2.4}$$

where $d_n^{(1)}$ is defined in (1.14) (see also (1.16)) and

$$v_n = M u_n - 1 \qquad (n \geqslant 0) \tag{2.5}$$

Moreover we have $v_n = M \sum_{l>n} h_l$, so that the sequence v_n is positive and decreases monotonically to 0.

Put first $1/2 \leqslant s < 1$. Then, according to (1.15), the term $d_0^{(1)}$ is finite and the power series $\sum_{n=0}^{\infty} d_n^{(1)} z^n$ is divergent at $z=1$. Thus, for these values of s, a direct application of (1.15) and the same Tauberian theorem for power series used above give $v_n \sim C_2 n^{1-(1/s)}$ and hence (a). Furthermore, using again (1.15), we have that if $1/(\ell+1) \leqslant s < 1/\ell$, with $\ell > 1$, then for $k \leqslant \ell$ the terms $d_0^{(k)}$ are finite and the power series $\sum_{n=0}^{\infty} d_n^{(\ell)} z^n$ is divergent at $z=1$. On the other hand, it is easy to check that under these circumstances (2.4) can be rewritten in the following way:

$$\sum_{n=0}^{\infty} v_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n = (z-1)^{\ell-1} \sum_{n=0}^{\infty} d_n^{(\ell)} z^n + \sum_{k=2}^{\ell} (z-1)^{k-2} \left(d_0^{(k-1)} + d_0^{(k)}\right) \tag{2.6}$$

so that the claimed result follows using the same reasoning as above, along with the positivity and monotonicity of the sequences $d_n^{(\ell)}$.

It remains to show that $z=1$ is a non-polar singularity for $\Phi(z)$. Now, from (1.15) we have that if $s \geqslant 1$, then $(1-z)\,\Phi(z) \to 0$ even though

$\Phi(z) \to \infty$ as $z \to 1_-$. Moreover, if $1/(\ell+1) \leqslant s < 1/\ell$, then, denoting by $H_\ell(z)$ the expression in (2.6), we have, again by (1.15), $(z-1)^\ell H_\ell(z) \to 0$ but $(z-1)^{\ell-1} H_\ell(z) \to \infty$ as $z \to 1_-$. The assertion then follows for each of these functions, and in particular for $\Phi(z)$. ∎

Let us now observe that the coefficients d_n can be considered as values of a function $d(x)$ when x ranges over the natural numbers. One may then examine the relation between the analytic properties of the function $d(x)$ determining the coefficients and those of the function defined by $D(z) = \sum_n d_n z^n$ (see for instance Dienes,[26] p. 335). Along these lines we now prove the following theorem.

Theorem 2.1 (Part Two). The function $\Phi(z)$ can be continued meromorphically to the entire z-plane with a branch cut along the ray $(1, +\infty)$. The meromorphic continuation is given by the formula, valid for any $\delta > 0$,

$$\Phi(z) = \frac{1}{(1-z)} \left(\frac{1}{2\pi i} \int_1^{+\infty} \int_{\mathrm{Re}\, x = \delta} d(x) \frac{t^{-x}}{t-z}\, dx\, dt \right)^{-1}$$

where $d(x) = (1+rx)^{-1/s}$.

Proof. The following proof relies on standard techniques of analytic continuation of power series based on the use of the Mellin transform. The first step in this approach is the construction of a function $d(x)$ defined on $\mathbb{R}_+$, which reproduces the numbers d_n at $x = n$ and extends to a function regular in the half-plane $\mathrm{Re}\, x > 0$. For our example this construction is effortless: $d(x) = (1+rx)^{-1/s}$. Nevertheless we shall sketch below a procedure which may be applied in more general situations, e.g. when the d_n's are not explicitly known. To this end, we first recall that $d_n = \psi_0^n(1)$, where ψ_0 is the inverse branch of f leaving fixed the origin. Let moreover $\psi: [0, 1] \to [0, q]$ a suitable smooth function which interpolates ψ_0 at those points: $\psi(d_n) = d_{n+1}$, so that $d_n = \psi^n(1)$ as well. Now, a standard method for dealing with the asymptotic behaviour of iterated functions starts considering the Abel equation (see, e.g., de Bruijn,[27] p. 160): $G(\psi(x)) = G(x) + 1$. If G is known, up to an additive constant, and ψ satisfies the above equation, one finds ψ^n by solving $G(\psi^n(x)) = G(x) + n$ for $\psi^n(x)$. Suppose one is able to determine a solution $G: [q, 1] \to [0, 1]$ of the Abel equation,[2] satisfying $G(1) = 0$ and $G(q) = 1$. Let $F(x) = G^{-1}(x)$:

[2] The problem of the existence of solutions of the Abel equation for a relatively large class of intermittent maps has been investigated by Prellberg.[6]

$[0, 1] \to [q, 1]$. A candidate for the function $d(x)$ is then obtained by extending $F(x)$ to $\mathbb{R}_+$ as follows:

$$d(x) = \psi^n(F(x-n)), \qquad n \leqslant x \leqslant n+1, \qquad n \geqslant 0$$

It is easy to check that in our case the function

$$\psi(x) := x(1+rx^s)^{-1/s} \tag{2.7}$$

satisfies the above requirement[3] and a real analytic solution G: $[q, 1] \to [0, 1]$ of the Abel equation (with ψ as in (2.7)) which satisfies $G(1) = 0$ and $G(q) = 1$ is given by $G(x) = (x^{-s} - 1)/r$, and its inverse is $F(x) = G^{-1}(x) = (1+rx)^{-1/s}$. Therefore we get $\psi^n(x) = F(G(x) + n) = x(1+nrx^s)^{-1/s}$ and $d(x)$ as announced above. Accordingly, the function $d(x)$ extends to a function regular in the half-plane $\operatorname{Re} x > 0$ and, for any $\delta > 0$,

$$d(x) \to 0, \qquad d'(x) = \mathcal{O}(x^{-1-(1/s)}), \qquad x \to \infty, \qquad \operatorname{Re} x \geqslant \delta$$

uniformly in $\arg x$. We can then proceed as in Evgrafov,[29] Section VII, Theorem 6.1. First, we take the Mellin transform of $d(-x)$,

$$d(-x) = \int_1^\infty w(t)\, t^x \frac{dt}{t}, \qquad w(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} x = -\delta} d(-x)\, t^{-x}\, dx$$

In the first expression we put $x = -n$ and multiply by z^n. Taking $|z|$ smaller than the distance from the origin to the contour $(1, \infty)$, that is $|z| < 1$, we sum over $n \geqslant 0$ and, as we may interchange the order of summation and integration, we get

$$\sum_{n=0}^{\infty} d_n z^n = \int_1^\infty \sum_{n=0}^{\infty} \frac{z^n}{t^{n+1}}\, w(t)\, dt = \int_1^\infty \frac{w(t)}{t-z}\, dt$$

The last integral converges uniformly in any closed region not containing points of the ray $(1, \infty)$. We finish the proof by inserting the representation of $w(t)$ in the above integral. ∎

Remarks. 1. If one wishes, one can investigate the behaviour of $\Phi(z)$ in a neighborhood of the branch point $z = 1$ with the help of the

[3] By the way, (2.7) is but the exact solution of the fixed point equation for the renormalization transformation with intermittency boundary conditions.[28]

above formula. For example, taking $s=1$ one finds that $\Phi(z)$ has a logarithmic branch point at $z=1$.

2. Consider the dynamical zeta function $\zeta(z)$ defined by the following formal series:[30]

$$\zeta(z)=\exp\sum_{n=1}^{\infty}\frac{z^n}{n}Z_n, \qquad Z_n=\sum_{x=f^n(x)}\prod_{k=0}^{n-1}\frac{1}{|f'(f^k(x))|}$$

It is an easy task to realize that $Z_n=1+\mathrm{tr}(P_N)^n$ provided $N>n$, where P_N is the $N\times N$ truncation of the transition matrix (1.5) and the 1 comes from the neutral fixed point. A staightforward algebraic calculation then gives

$$\begin{aligned}(1-z)/\zeta(z)&=\lim_{N\to\infty}\exp -\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{tr}(P_N)^n\\ &=\lim_{N\to\infty}\det(I-zP_N)\\ &=1-\sum_{n=1}^{\infty}\rho_n z^n\end{aligned}$$

and therefore

$$\zeta(z)=(1-z)^{-1}\,\Phi(z)$$

The above identity and Theorem 2.1 (Part two) answer the question raised by Dalqvist[31] for this particular model (see also Rugh[32] for related results on Fredholm determinants).

3. SCALING AND MIXING RATES

We now briefly dwell upon some consequences of Theorem 2.1 (Part one). Given $U\in L^2([0,1],\mathscr{B},\nu)$ one may consider the formal power series $S_U(z)$ given by

$$S_U(z):=\sum_{n=0}^{\infty}z^n\nu(U\cdot U\circ f^n)$$

Take first $U=\chi_{A_1}$, the indicator function of the interval A_1. From the previous section we have that $\nu(\chi_{A_1}\cdot\chi_{A_1}\circ f^n)=\nu(A_1\cap f^{-n}A_1)=u_n$. Therefore

$$S_{\chi_{A_1}}(z)=\Phi(z) \tag{3.1}$$

3.1. Mixing Rate When the Invariant Measure is Finite

In this subsection we shall assume that $s<1$ (so that $M<\infty$). We can then consider the generating function of the auto-correlation function of probability measure $\mu=\nu/M$ for the observable χ_{A_1}. An easy calculation using (3.1) shows that

$$\sum_{n=0}^{\infty} z^n \left(\mu(A_1 \cap f^{-n}A_1)-(\mu(A_1))^2\right)=(\mu(A_1))^2 \cdot \sum_{n=0}^{\infty} v_n z^n \tag{3.2}$$

where $\mu(A_1)=1/M$ and the v_n's are defined in statement (a) of Theorem 2.1 (Part one). The asymptotics of v_n given there and (3.2) yield at once

$$\mu(A_1 \cap f^{-n}A_1)-(\mu(A_1))^2 \sim C_2(\mu(A_1))^2\, n^{1-(1/s)} \tag{3.3}$$

Next, given $k\in\mathbb{Z}^+$, $k>1$, the analogous of relation (2.2) for $u_n^{(k)}:=\nu(A_k\cap f^{-n}A_k)$ reads $u_n^{(k)}=0$ for $0<n<k$, and

$$\frac{u_n^{(k)}}{\nu(A_k)}=u_0\rho_n+\cdots+u_{n-k}\rho_k, \qquad n\geqslant k \tag{3.4}$$

Therefore we have

$$\frac{1}{\nu(A_k)}\sum_{n=k}^{\infty} z^n \nu(A_k\cap f^{-n}A_k)=\Phi(z)\cdot\sum_{n=k}^{\infty} z^n\rho_n \tag{3.5}$$

From this, we obtain the following expression for the generating function of the correlation function for the indicator function χ_{A_k}:

$$\sum_{n=k}^{\infty} z^n \left(\mu(A_k\cap f^{-n}A_k)-(\mu(A_k))^2\right)$$
$$=(\mu(A_k))^2\left[\frac{1}{\mu(A_k)}\,\Phi(z)\cdot\sum_{n=k}^{\infty} z^n\rho_n-\sum_{n=k}^{\infty} z^n\right] \tag{3.6}$$

The term between square brackets can be decomposed as $S_k(z)+R_k(z)$ with

$$S_k(z)=\frac{\sum_{n=k}^{\infty} z^n\rho_n}{\nu(A_k)}\cdot\sum_{n=0}^{\infty} z^n v_n \tag{3.7}$$

and

$$R_k(z) = \frac{\sum_{n=k}^{\infty} z^n \rho_n}{\nu(A_k)(1-z)} - \sum_{n=k}^{\infty} z^n = -\frac{1}{\nu(A_k)} \sum_{n=k}^{\infty} d_n z^n \tag{3.8}$$

We now use the following result, whose proof can be found in Chung,[33] Chap. I.5, Lemma A.

Lemma 3.1. Let $\{s_n\}_{n \geqslant 0}$ be a sequence of nonnegative numbers not all vanishing. If

$$\lim_{n \to \infty} \frac{s_n}{\sum_{m=0}^{n} s_m} = 0$$

then, whenever the sequence $\{t_n\}_{n \geqslant 0}$ of real numbers has a limit, we have

$$\lim_{n \to \infty} \frac{\sum_{m=0}^{n} s_m t_{n-m}}{\sum_{m=0}^{n} s_m} = \lim_{n \to \infty} t_n$$

Using this result with $t_m = v_m$ and $s_m = \rho_{k+m}$ we see that the coefficient of z^n (with $n \geqslant k$) in $S_k(z)$ is asymptotically equivalent to

$$\left(\frac{\sum_{m=0}^{n-k} \rho_{k+m}}{\nu(A_k)} \right) \cdot v_{n-k} \sim v_n \tag{3.9}$$

where (1.8) has been used and the last asymptotic equivalence holds for each fixed $k \in \mathbb{Z}^+$. Putting together (3.6)–(3.9) along with statement (a) of Theorem 2.1 (Part one) and (1.15) we get,

$$\mu(A_k \cap f^{-n} A_k) - (\mu(A_k))^2 \sim C_2(\mu(A_k))^2 \, n^{1-(1/s)} \tag{3.10}$$

We now consider an arbitrary Borel subset $E \subseteq A_k$, with $m(E) > 0$. We have $\nu(E \cap f^{-n} E) = 0$ for $0 < n < k$, and

$$\frac{\nu(E \cap f^{-n} E)}{\nu(E)} = u_0 m(f^{-(n-k)}(E) \cap A_n) + \cdots + u_{n-k}\, m(E) \tag{3.11}$$

for $n \geqslant k$, because whenever $F \subseteq B_l$ we have $\nu(F) = m(G)$ with $G = f(F) \subseteq A_l$. Therefore, the generating function of the correlation function for the indicator function χ_E is exactly the same as (3.6) provided A_k is replaced by E and ρ_n by $m(f^{-(n-k)}(E)\cap A_n)$ $(n\geqslant k)$. The same reasoning as above along with the fact that $\nu(E)=\sum_{l\geqslant 0} m(f^{-l}(E)\cap A_{k+l})$ then gives again

$$\mu(E\cap f^{-n}E)-(\mu(E))^2\sim C_2(\mu(E))^2\, n^{1-(1/s)} \tag{3.12}$$

In an entirely analogous way one shows that (3.12) holds true for $E\subseteq\bigcup_{l\in J} A_l$ where $J\subset\mathbb{Z}^+$ is any given finite set.

Let $\mathscr{B}$ be the Borel σ-algebra on $[0,1]$. Given $E\subset\mathscr{B}$, we define the *mixing rate* $\mu_n(E)$ of E as

$$\mu_n(E):=\frac{\mu(E\cap f^{-n}E)-(\mu(E))^2}{(\mu(E))^2} \tag{3.13}$$

The mixing rate is not uniform in $E\subset\mathscr{B}$. For instance, (3.10) follows from (3.6)–(3.8) for each fixed $k\in\mathbb{Z}^+$, but not uniformly in k. To recover uniformity, we define

$$B_+:=\bigcup_{\varepsilon}\{E\in\mathscr{B}: m(E)>0,\ E\subseteq[0,1]\backslash(0,\varepsilon)\} \tag{3.14}$$

An easy consequence of the above discussion is the following result:

Lemma 3.2. Let $E, F\in B_+$. Then $\mu_n(E)\sim\mu_n(F)$.

Therefore, one can define the (self-) mixing rate $\mu_n(f)$ of the map f as the rate of asymptotic decay of the sequences $\{\mu_n(E)\}$, with $E\in B_+$. We summarize the above in the following

Theorem 3.3. If $M<\infty$ then $\mu_n(f)=C_2 n^{1-(1/s)}$.

Remark 1. Let $E\subset\mathscr{B}$, with $0<m(E)<1$, and $E^c=[0,1]\backslash E$. The following identity holds plainly

$$\frac{\mu(E^c\cap f^{-n}E^c)}{\mu(E^c)}+\frac{\mu(E^c\cap f^{-n}E)}{\mu(E^c)}=1$$

Moreover, μ being f-invariant, we also have

$$\frac{\mu(E\cap f^{-n}E)}{\mu(E)}+\frac{\mu(E^c\cap f^{-n}E)}{\mu(E)}=1$$

One may then subtract $\mu(E)+\mu(E^c)=1$ from these two relations, multiply the resulting identities by $\mu(E)$ and $\mu(E^c)$, respectively, and compare the results. One obtains

$$\mu(E\cap f^{-n}E)-(\mu(E))^2=\mu(E^c\cap f^{-n}E^c)-(\mu(E^c))^2$$

Now assume that $E\subset B_+$ so that $E^c\not\subset B_+$. The above identity, Lemma 3.2 and Theorem 3.3 then imply that

$$\mu_n(E^c)=\left(\frac{\mu(E)}{\mu(E^c)}\right)^2 C_2 n^{1-(1/s)}, \qquad E\subset B_+, \qquad 0<m(E)<1$$

This can be used, for instance, to evaluate the mixing rate of the set $D_k=\bigcup_{l>k} A_l$, for any $k\in\mathbb{Z}^+$.

Remark 2. We point out that Theorem 3.3 gives the exact rate of mixing of the map *f*, not just a bound for it. In particular they improve all previously known bounds.[4, 5] The above results can be viewed as statements about the decay of correlations for test functions as simple as indicators of sets in B_+. This makes the mixing rate (as defined above) determined by nothing but the distribution of return times: $\nu\{x\in X:\tau(x)>n\}$ (compare (1.16)). On the other hand, when dealing with correlation functions of a broader class of observables, one expects a richer behaviour depending also of the smoothness properties of the functions involved. In particular one may obtain faster decays. We refer to Isola,[14] Liverani *et al.*[15] and Young[16] for different approaches yielding more general results.

3.2. Scaling Rate, Wandering Rate, and Return Sequence When the Invariant Measure Is Infinite

When $M=\infty$, given $E\subset\mathscr{B}$, with $\nu(E)>0$, we can define the *scaling rate* $\sigma_n(E)$ of E as

$$\sigma_n(E):=\frac{\nu(E\cap f^{-n}E)}{(\nu(E))^2} \tag{3.15}$$

Clearly we have $\sigma_n(A_1)=u_n$. In addition, for any given $k\in\mathbb{Z}^+$, using (3.5) and Lemma 3.1 we have,

$$\nu(A_k\cap f^{-n}A_k)\sim\nu(A_k)\cdot\left(\sum_{m=0}^{n-k}\rho_{k+m}\right)\cdot u_{n-k}\sim(\nu(A_k))^2\cdot u_n \tag{3.16}$$

More generally, reasoning as in the previous subsection, one shows that $\sigma_n(E) \sim u_n$ for all $E \in B_+$. We then define the scaling rate $\sigma_n(f)$ of the map f as the rate of asymptotic decay of the sequences $\{\sigma_n(E)\}$, $E \in B_+$. By virtue of statement (b) in Theorem 2.1 (Part one) we obtain

Theorem 3.4. If $M = \infty$ we have

$$\sigma_n(f) = \begin{cases} C_1 n^{-1+(1/s)}, & \text{if} \quad s > 1 \\ 1/\log n, & \text{if} \quad s = 1 \end{cases}$$

From the scaling rate, defined above, one can compute some other natural objects arising in the ergodic theory of transformations preserving infinite measures,[4] notably the *wandering rate* $w_n(f)$ and the *return sequence* $r_n(f)$. The wandering rate $w_n(E)$ of a Borel set E is defined by $w_n(E) = \nu(\bigcup_{k=0}^{n-1} f^{-k}E)$. The asymptotic equivalence of wandering rates for sets in B_+ has been proved in Thaler,[12] Theorem 3. Thus one defines $w_n(f)$ as the rate of growth of the sequences $\{w_n(E)\}$, $E \in B_+$. In our case this is simply given by the partial sums $\sum_{k=0}^{n} d_k$. On the other hand, the existence of $r_n(f)$ is what makes the transformation f *pointwise dual ergodic*,[10] namely such that $(1/r_n) \sum_{k=0}^{n-1} P^k U \to e \cdot m(U)$ for any $U \in L^1([0, 1], \mathscr{B}, \nu)$. Here P is the operator acting on $L^1([0, 1], \mathscr{B}, \nu)$ dual to f, that is satisfying $\int PU \cdot V dm = \int U \cdot V \circ f \, dm$. Notice however that this property does not imply that the partial averages $(1/r_n) \sum_{k=0}^{n-1} U \circ f^k$ converge m-almost surely to the number $\nu(U)$. On the contrary, it can be proved[10, 11] that this cannot hold, not even for one particular sequence of constants r_n. Nevertheless, if the sequence r_n is (asymptotically equivalent to) the return sequence, then these partial averages converge in measure to $\nu(U)$.[10, 11] Now, such quantities can be readily obtained putting together Theorem 3.4 and the asymptotic equivalences[10]:

$$w_n(f) \sim \frac{n}{\sum_{k=0}^{n} \sigma_k}, \qquad r_n(f) \sim \sum_{k=0}^{n} \sigma_k \tag{3.17}$$

4. CONCLUDING REMARKS

We finally point out that using the above and results from Feller[25] one can obtain several limit theorems, at least for observables such as indicator functions of sets in B_+. To give an example where Feller results are directly applicable, consider the test function $U = \chi_{A_1}$. Then $N_n(x) := U(x) + \cdots + U(f^{n-1}(x))$ gives the number of passages in the

[4] For a general survey on infinite ergodic theory see Aaronson.[34]

state 1 up to the nth iterate of the map f. Let moreover $g: X \to X$ be the induced map defined by $g(x) = f^{\tau(x)}(x)$. Then $S_n(x) := \tau(x) + \tau(g(x)) \cdots + \tau(g^{n-1}(x))$ is the total number of iterates of f needed to observe n passages in the state 1. The relation between u_n defined in Section 1 and the above quantities can be obtained as follows: first notice that $(N_n = k) = (S_k \leqslant n < S_{k+1}) = (S_k \leqslant n) - (S_{k+1} \leqslant n)$. Thus $m(N_n = k) = m(S_k \leqslant n) - m(S_{k+1} \leqslant n)$, which is the same as $m(S_k \leqslant n) = \sum_{r=k}^{n} m(N_n = r)$. Moreover $m(S_k = n) = m(S_k \leqslant n) - m(S_k \leqslant n-1)$ for $k < n$ and $m(S_n = n) = m(S_n \leqslant n)$. Therefore we have the following chain of identities

$$
\begin{aligned}
u_n &= \sum_{k=1}^{n} m(S_k = n) = \sum_{k=1}^{n} m(S_k \leqslant n) - \sum_{k=1}^{n-1} m(S_k \leqslant n-1) \\
&= \sum_{k=1}^{n} \sum_{r=k}^{n} m(N_n = r) - \sum_{k=1}^{n-1} \sum_{r=k}^{n-1} m(N_{n-1} = r) = m(N_n) - m(N_{n-1})
\end{aligned}
$$

where $m(N_n)$ denotes the mean of the random variable N_n (set $N_0 = 0$). Thus, u_n may be regarded as the expected number of passages in the state 1 (renewals) per iteration of the map f (after $n-1$ iterations). The above identity relates (2.1) with the standard renewal equation[35] for $m(N_n)$. Take first $s < 1/2$. Then, using the notation of Section 1, we have $\sigma^2 := M^{(2)} - M^2 < \infty$. This implies that the associated Markov chain has ergodic degree at least two. One then shows[25] that the mean and the variance of N_n are asymptotically equal to n/M and $\sigma/M^{3/2}$. Observing that $m(N_n \geqslant k) = m(S_k \leqslant n)$, one obtain the following (central) limit theorem:

$$
m\left(x \in X : N_n(x) \geqslant \frac{n}{M} - \frac{\sigma \sqrt{n}}{M^{3/2}} \alpha \right) \to \frac{1}{2\pi} \int_{-\infty}^{\alpha} e^{-y^2/2} \, dy
$$

In the case $1/2 \leqslant s < 1$, in which the associated Markov chain has ergodic degree one, as well as in the null recurrent case ($s \geqslant 1$), one obtains different, non-normal, limiting distributions, for which we refer to Feller's paper.

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